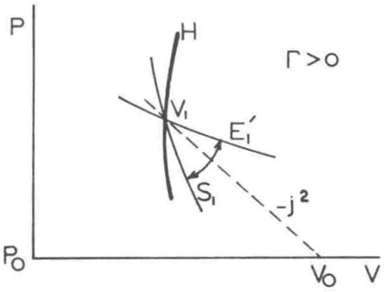

FIG. 15. Paths in P-V plane for $\Gamma>0$. Adiabatic expansion from V_1 takes place on path between S_1 and E_1'. Hugoniot curve H, excluded from this region.

We can illustrate this restriction by means of a P-V diagram as shown in Fig. 15, for the case $\Gamma>0$. On the equilibrium surface we have

$$\left(\frac{\partial P}{\partial V}\right)_{E'}=\left(\frac{\partial P}{\partial V}\right)_s+\left(\frac{\partial P}{\partial S}\right)_V\left(\frac{\partial S}{\partial V}\right)_{E'}$$

$$=\left(\frac{\partial P}{\partial V}\right)_s+\frac{\Gamma}{V}(P-P_0)\ .$$

Hence, for $\Gamma>0$ and $P>P_0$ the curve of constant E' lies above the isentrope S as shown, and adiabatic fluctuations consistent with Ineq. (27) lie between these curves. When $\Gamma<0$, the relative positions are reversed. For stable shocks the Hugoniot curve is excluded from the region bounded by these curves.

We can also consider the stability problem from the point of view of the restoring forces invoked during a virtual displacement. Returning to Eqs. (28) and (29) and retaining only first order terms in an expansion about a Hugoniot state, specified by $P=P_1$, $V=V_1$, gives

$$\sigma=P_1+\left[\left(\frac{2}{V_0-V_1}\right)\frac{dE'}{dV}+j^2\right]dV+\cdots\ , \tag{33a}$$

$$=P_1+\left[\left(\frac{2}{V_0-V_1}\right)T_1\frac{dS}{dV}-j^2\right]dV+\cdots\ . \tag{33b}$$

The paths along which the derivatives are taken is so far arbitrary. [In Eqs. (30) and (31) we also specified $d\sigma/dV=dP/dV$.]

Expressions analogous to Eq. (33) can be written for the equilibrium surface; thus,

$$P=P_1+\left[\left(\frac{\partial P}{\partial E'}\right)_V\frac{dE'}{dV}+\left(\frac{\partial P}{\partial V}\right)_{E'}\right]dV+\cdots$$

$$=P_1+\left[\frac{\Gamma_1}{V_1}\frac{dE'}{dV}+\left(\frac{\partial P}{\partial V}\right)_{E'}\right]dV+\cdots \tag{34a}$$

$$=P_1+\left[\frac{\Gamma_1T_1}{V_1}\left(\frac{dS}{dV}\right)+\left(\frac{\partial P}{\partial V}\right)_s\right]dV+\cdots\ . \tag{34b}$$

The difference between Eq. (33) and Eq. (34) is

$$\sigma-P=\left[\left(\frac{2}{V_0-V_1}\right)(1-a_1)\frac{dE'}{dV}+j^2-\left(\frac{\partial P}{\partial V}\right)_{E'}\right]dV, \tag{35a}$$

or

$$\sigma-P=\left[\left(\frac{2}{V_0-V_1}\right)(1-a_1)T_1\frac{dS}{dV}-j^2-\left(\frac{\partial P}{\partial V}\right)_s\right]dV, \tag{35b}$$

where the definition of "a," Eq. (15), has been used.

We now consider fluctuations in volume δV consistent with the thermodynamic restriction, Ineq. (26) and require for stability that the restoring force be opposed to the displacement, i.e.,

$$(\sigma-P)\delta V\geq 0\ , \tag{36}$$

for stability. Along the two bounding curves, $dE'=0$ and $dS=0$, this implies

$$[j^2-(\partial P/\partial V)_{E'}](\delta V)^2\geq 0\ ,$$

and

$$[-j^2-(\partial P/\partial V)_s](\delta V)^2\geq 0\ .$$

Thus,

$$(\partial P/\partial V)_{E'}\leq j^2\ , \tag{37a}$$

and

$$(\partial P/\partial V)_s\leq -j^2\ . \tag{37b}$$

These restrictions are shown in Fig. 16.

For intermediate paths between these bounds, Ineq. (36) stipulates, from Eq. (35),

$$\left(\frac{2}{V_0-V_1}\right)(1-a_1)T_1\frac{dS}{dV}-j^2-\left(\frac{\partial P}{\partial V}\right)_s$$

$$=\left(\frac{2}{V_0-V_1}\right)(1-a_1)\frac{dE'}{dV}+j^2-\left(\frac{\partial P}{\partial V}\right)_{E'}\geq 0\ . \tag{38}$$

The former of these is clearly satisfied, provided Ineqs. (37) are valid, for $a<1$ and $dS/dV\geq 0$; the latter when $a>1$ and $dE'/dV\leq 0$. Thus, stability with respect to all admissible fluctuations consistent with Ineq. (26) is guaranteed by Ineqs. (37).

The restrictions on the slope of the Hugoniot P-V curve implied by Ineqs. (37) can be derived from Eq. (13). We note that Ineq. (37b) is just the subsonic condition,

$$M^2=-j^2(dV/dP)_s\leq 1\ ,$$

and this restriction implies, for $M^2a<1$,

$$-1\leq j^2(dV/dP)_H\ ,$$

as shown in Fig. 1.

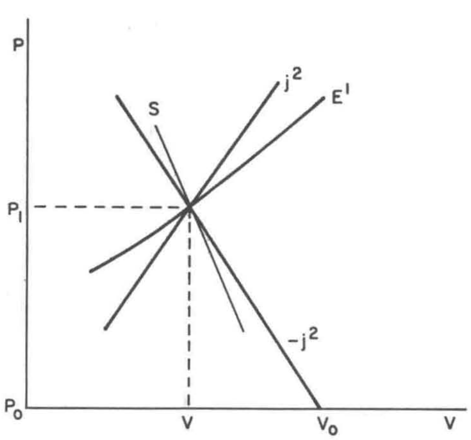

FIG. 16. Relative positions of Rayleigh line, $-j^2$, reflected Rayleigh line, j^2, isentrope, S, and isoenergetic line, E', for stable shocks.

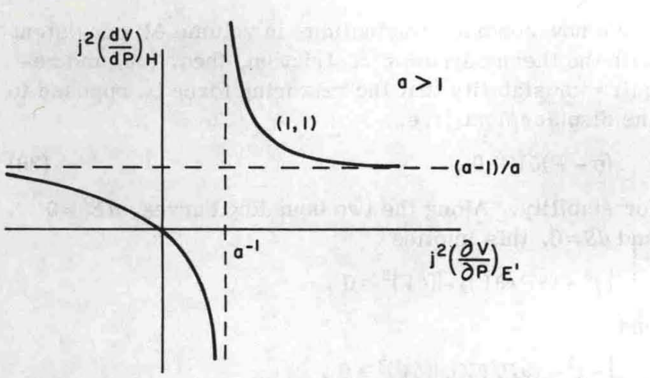

FIG. 17. Plot of $j^2\ (dV/dP)_H$ as function of $j^2\ (\partial V/\partial P)_{E'}$ when $a>1$. Stable case corresponds to $j^2\ (\partial V/\partial P)_{E'}>1$, and $j^2 \times (dV/dP)_H<1$.

With the substitutions

$$\left(\frac{\partial P}{\partial V}\right)_{E'}=\left(\frac{\partial P}{\partial V}\right)_s+\frac{\Gamma(P-P_0)}{V}$$

$$=\left(\frac{\partial P}{\partial V}\right)_s+2aj^2 \qquad (\sigma=P),$$

Eq. (16) can be written

$$j^2\left(\frac{dV}{dP}\right)_H=\frac{j^2(\partial V/\partial P)_{E'}(a-1)}{aj^2(\partial V/\partial P)_{E'}-1}. \tag{39}$$

A plot of this function for the case $a>1$ which, excluding the region $M^2a>1$, is the only remaining case of interest, is shown as Fig. 17. From this figure it is clear that the restriction, Ineq. (37a), also implies the inequality

$$j^2(dV/dP)_H\leq 1\ .$$

We therefore conclude that the stability condition, Ineq. (36), when combined with the thermodynamic restriction, Ineq. (27), implies the criterion for shock stability

$$-1\leq j^2(dV/dP)_H\leq 1$$

in agreement with earlier arguments.

To complete the theory we must include the other well-known condition for stability, namely, that the shock travel with supersonic velocity with respect to the undisturbed medium ahead of the shock. This has been shown elsewhere.[9] Moreover, we still have to consider the branch 3*b* of Fig. 1, for which $M^2a>1$.

From Eq. (14), with $P=P_0$, $V=V_0$, it is clear that the Hugoniot and isentrope have the same slope at the initial state. Hence, the supersonic condition

$$M_0^2=-j^2(\partial V/\partial P)_s>1$$

also implies

$$j^2(dV/dP)_H<-1 \tag{40}$$

in the initial state.

Now consider Hugoniot curves of two different types that are assumed to lie on branch 3*b* of Fig. 1 as illustrated in Fig. 18. A Hugoniot curve of type I, that approaches the shocked state 1 from below the Rayleigh line can be ruled out on the basis that one or the other of the limits of Ineq. (19) would be exceeded before the slope of the Hugoniot could take on values pertaining to the region in question, i.e., $j^2(dV/dP)_H<-1$. Alternatively, a Hugoniot of type II necessarily crosses the Rayleigh line at a lower pressure as at point 2. This state, however, is a thermodynamic equilibrium state and could therefore be considered an initial state for the shock transition from 2 to 1. However, according to Ineq. (40), the supersonic condition would be violated. We conclude, therefore, that the branch 3*b* of Fig. 1, for which $M^2a>1$, is unattainable.

TABLE I. Limits of various derivatives for stable shocks.

$-1\leq j^2\ (dV/dP)_H\leq 1$
$(\partial P/\partial V)_S\leq -j^2$
$(\partial P/\partial V)_{E'}\leq j^2$
$0\leq T\ (dS/dP)_H\leq (V_0-V)$
$0\leq (dE'/dP)_H\leq (V_0-V)$
$0\leq (dU/dP)_H\leq U/(P-P_0)$
$0\leq (du/dP)_H\leq u/(P-P_0)$
$(dS/dE')_H\geq 0$

The symmetry of Ineq. (19) is reflected in other equivalent relations derived by substituting from the jump conditions, Eqs. (1)–(3). These are shown in Table I. We note that one consequence is that the shock velocity is a monotonic function of the particle velocity.

Let us now consider further the consequences of violation of each of the limits of Ineq. (19). Figure 19 shows a case in which the lower limit is violated between points A and C. From Eq. (31) it is seen that the entropy along the Hugoniot curve is a maximum at A and a minimum at B with respect to neighboring Hugoniot states. If we plot entropy as a function of pressure along the Hugoniot, we get a curve like that in Fig. 20.

From our criteria we deduce that shock waves whose final states fall within A–C are unstable. If the final pressure falls within this range, a two-wave configuration is produced in which the first wave carries the material to state A, and a subsequent shock with initial state A carries the material to higher pressure, less than C; if the final pressure exceeds C, a single shock is again stable. Instabilities of this type and the two-shock configuration have been widely observed.[12] It is important to notice that under these conditions the pres-

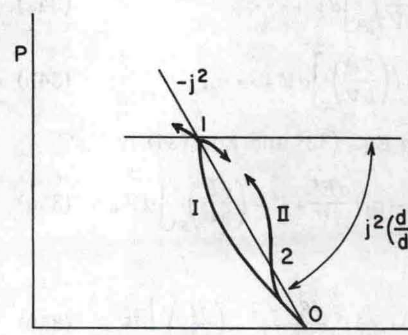

FIG. 18. Unstable Hugoniot curves for which $j^2\ (dV/dP)_H<-1$ at point 1.